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Large-scale modulations of edge waves

By T. R. AKYLAS
Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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The temporal and spatial evolution of large-scale modulations of weakly nonlinear edge waves on a uniformly sloping beach is studied using the full water-wave formulation for beach angles $\alpha = \pi/2N$. Equations governing the evolution of envelopes of edge waves, excited by resonant interactions with incident wavetrains, are derived. It is deduced that a uniform train of free periodic edge waves is always unstable to large-scale variations, so that envelope solitons will develop; the resulting three-dimensional solitons are described in detail. In addition, it is shown that steady-state standing subharmonic edge waves, excited by incident wavetrains on a long, mildly sloping beach, can be unstable to large-scale modulations. The possible physical significance of these findings is discussed.

1. Introduction

Edge waves are trapped wave modes, propagating along the coastline of a sloping beach, and decaying in the offshore direction. Although the existence of a solution of the linear water-wave problem, representing edge waves on a uniformly sloping beach, has been known for quite some time (Stokes 1846), the physical significance of edge waves has been only recently recognized (Munk, Snodgrass & Carrier 1956). At present, it is generally believed that edge waves play an important role in the formation of coastal patterns (Guza & Inman 1975), and in the generation of rip currents and periodic circulation cells in the nearshore region (Bowen & Inman 1969).

The main reason, for which very little physical relevance has been attached to edge waves for a long time, stems from the fact that the appropriate generation mechanisms of such trapped wave disturbances are not immediately clear. Greenspan (1956) first demonstrated that edge waves can be excited by atmospheric forcing due to storms moving along the coastline. Later, Guza & Davis (1974), in an attempt to explain the experimental observations of Galvin (1965) and Bowen & Inman (1969), pointed attention to the fact that edge waves can also be generated by nonlinear interactions with incoming wavetrains: using the shallow-water approximation, they showed that a monochromatic harmonic wavetrain of frequency ω, normally incident and reflected on a beach, is unstable to subharmonic standing edge-wave perturbations of frequency $\frac{1}{2}\omega$. Guza & Bowen (1976) and, more recently, Rockliff (1978) studied the full nonlinear interaction of subharmonic edge waves with incident wavetrains in the shallow-water approximation, and concluded that the initial instability eventually gives rise to steady-state edge waves. In the further development of the theory, Whitham (1976) pointed out that the shallow-water approximation is inadequate to describe the behaviour of edge waves uniformly in the offshore direction; using the full water-wave formulation, he calculated the leading-order nonlinear correction to the linear dispersion relation of travelling Stokes edge waves, and thereby deduced that periodic finite-amplitude edge waves are always unstable to large-scale modulations. Later, Minzoni & Whitham (1977) studied the excitation of monochromatic subharmonic edge waves by incident wavetrains, using the full water-wave formulation; in the small-beach-angle limit their results confirm the predictions of the shallow-water theory.

In the present paper, the theory of Whitham (1976) and Minzoni & Whitham (1977) is extended to interacting edge-wave packets. Using the full water-wave formulation, evolution equations for the envelopes of subharmonic edge-wave packets, excited by incident wavetrains, are derived. The temporal and spatial non-uniformities encountered in the weakly nonlinear regime are handled using the method of multiple scales. The resulting uniformly valid envelope equations are employed to study two problems of physical interest: first, the modulational instability of free periodic travelling edge waves is re-examined, and it is confirmed that the instability, predicted by Whitham (1976), indeed leads to a series of envelope solitons; in addition, the present formulation enables us to describe the three-dimensional structure of edge-wave envelope solitons. Secondly, the stability to large-scale modulations of the steady-state subharmonic edge waves, predicted by Guza & Bowen (1976) and Minzoni & Whitham (1977), is investigated. It is found that, in the small-beach-angle limit, large-scale modulations may grow, the instability increasing with the amplitude of the incoming wave train. Finally, the possible physical significance of this instability is discussed briefly.

2. Formulation

Consider a uniformly sloping beach of angle α. A coordinate system is chosen such that x, y and z are the offshore, vertical and longshore coordinates respectively. The classical gravity water-wave problem can be formulated in terms of a velocity potential $\Phi(x,y,z,t)$ and the free-surface elevation $y = \eta(x,z,t)$. It proves convenient to use dimensionless (primed) variables:

$$x = lx', \quad y = ly', \quad z = lz', \quad T = \left(\frac{l}{g}\right)^{\frac{1}{2}} t', \quad \eta = a\eta', \quad \Phi = a(lg)^{\frac{1}{2}}\Phi',$$

where g is the gravitational acceleration, l is a typical perturbation wavelength, and a denotes a typical perturbation amplitude. When the primes are dropped, the problem, in terms of dimensionless variables, consists of Laplace's equation for the velocity potential in the wedge $-x\tan\alpha < y < \epsilon\eta$, subject to the (nonlinear) kinematic and pressure conditions at the free surface $y = \epsilon\eta$, and the bottom boundary condition at $y = -x\tan\alpha$. The dimensionless parameter $\epsilon = a/l$ is a measure of nonlinearity, and, for the purpose of the subsequent theoretical development, it will be assumed small, $\epsilon \ll 1$. It is then appropriate to expand the free-surface conditions about the fixed level $y = 0$, so that the water-wave problem, expressed in terms of the velocity potential alone, and correct to $O(\epsilon^2)$, reads

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \quad (-x\tan\alpha < y < 0), \tag{1}$$

$$\begin{aligned} \Phi_y + \Phi_{tt} = {} & -\epsilon|\nabla\Phi|^2_t + \epsilon\{\Phi_t(\Phi_y + \Phi_{tt})\}_y - \tfrac{1}{2}\epsilon^2\{\Phi_x|\nabla\Phi|^2_x + \Phi_y|\nabla\Phi|^2_y + \Phi_z|\nabla\Phi|^2_z\} \\ & + \epsilon^2\{\Phi_t|\nabla\Phi|^2_t\}_y + \tfrac{1}{2}\epsilon^2\{(\Phi_y + \Phi_{tt})(|\nabla\Phi|^2 - \Phi_t^2)\}_y \quad (y = 0), \end{aligned} \tag{2}$$

$$\Phi_x \sin\alpha + \Phi_y \cos\alpha = 0 \quad (y = -x\tan\alpha). \tag{3}$$

The linear theory is obtained by setting $\epsilon = 0$ in (2), and has been the subject of numerous investigations. (A detailed account of these studies is given by Whitham (1979).) For our purposes, it is sufficient to quote that the linearized problem (1)–(3) ($\epsilon = 0$) admits two distinct types of time-harmonic wave solutions: first, a discrete spectrum of edge-wave modes, which propagate along the shore and decay seawards, and, secondly, a continuous spectrum of wavetrains incident from deep water and reflected on the beach. In particular, the Stokes edge-wave mode is possible for $\alpha < \frac{1}{2}\pi$ and corresponds to the potential

$$\Phi = F(x,y)\,\mathrm{e}^{\mathrm{i}(kz-\omega t)} + \text{c.c.}, \tag{4}$$

where

$$\omega^2 = k \sin\alpha, \tag{5}$$

$$F(x,y) = \exp\left(-kx\cos\alpha + ky\sin\alpha\right), \tag{6}$$

and c.c. denotes the complex conjugate. Wave disturbances in the form of periodic wavetrains, incident and reflected on a beach, are described by

$$\Phi = f(x,y)\,\mathrm{e}^{\mathrm{i}(kz-\omega t)} + \text{c.c.}, \tag{7}$$

with $\omega^2 > k$. The specific form of $f(x,y)$ depends on the wave-reflection properties of the beach, which must be determined experimentally. The present discussion will be confined to normally incident, perfectly reflected wavetrains, so that Φ corresponds to a uniform standing wavetrain in deep water, and k is taken to be zero in (7); under these conditions, an explicit formula for $f(x,y)$ can be derived (see Whitham 1979).

According to the linear theory, edge waves and incoming wavetrains on a beach can coexist, without any interaction. However, it is well known that, in the weakly nonlinear regime ($0 < \epsilon \ll 1$), significant energy exchanges between different wave modes can occur, if certain resonance conditions are satisfied. In the following sections, nonlinear evolution equations are derived for the envelopes of two subharmonic edge-wave packets, excited by an incident wavetrain owing to nonlinear resonant interactions.

3. Evolution of the envelopes at the shoreline

Consider a normally incident wavetrain of frequency ω interacting with two Stokes edge-wave packets of frequency $\frac{1}{2}\omega$ propagating in opposite directions along a beach. The nonlinear parameter ϵ is based on the amplitude of the incident wavetrain in deep water, and the slopes of the edge-wave packets are taken to be $O(\epsilon^{\frac{1}{2}})$. Such a balance ensures that the nonlinear interactions between the edge waves themselves occur on the same timescale as the interactions between incident and edge-wave packets. Accordingly, a suitable expansion of the velocity potential for the three interacting wave packets is

$$\Phi(x,y,z,t) = \epsilon^{-\frac{1}{2}} F(x,y)\{(A_+(X,Y,Z,T)\,\mathrm{e}^{\mathrm{i}\theta_+} + \text{c.c.}) + (A_-(X,Y,Z,T)\,\mathrm{e}^{\mathrm{i}\theta_-} + \text{c.c.})\} \\ + \{S(x,y;X,Y,Z,T)\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}\} + \dots, \tag{8}$$

where

$$\theta_\pm = \pm kz - \tfrac{1}{2}\omega t, \quad \omega^2 = 4k\sin\alpha, \tag{9}$$

and

$$X = \mu x, \quad Y = \mu y, \quad Z = \mu z, \quad T = \mu t, \quad \mu \ll 1 \tag{10}$$

are long scales associated with the evolution of the wave envelopes. The $O(1)$ term in (8) represents the incident and reflected wavetrain, and will be made specific later.

In addition, the dependence on the long scales remains unspecified at this stage; it will be determined by the requirement that (8) is uniformly valid in space and time.

The nonlinear equations that govern the temporal and spatial evolution of the edge-wave envelopes A_+ and A_- can be derived by substituting the proposed expansion (8) into (1)–(3), and solving perturbatively to the required order of approximation. However, using already known results, it is possible to obtain some information about the appropriate envelope equations, without going through the formal analysis in detail.

As is well known, in simpler problems, where there is no dependence on the offshore coordinate, the envelope $A(Z,T)$ of a finite-amplitude travelling wave packet $A(Z,T)\,\mathrm{e}^{\mathrm{i}(kz-\Omega t)}$ satisfies the nonlinear Schrödinger equation

$$A_T + c_g A_Z = \tfrac{1}{2}\mathrm{i}\mu\Omega''\,A_{ZZ} + \mathrm{i}\frac{\epsilon}{\mu}\Omega_2 A^2 A^*, \tag{11}$$

where $\Omega = \Omega(k)$ is the linear dispersion relation, $c_g = \Omega'(k)$ is the group velocity, Ω_2 is a certain real coefficient depending on the particular problem, and the amplitude of the wave packet is $O(\epsilon^{\frac{1}{2}})$. The origin of the various terms in (11) can be easily identified: the linear terms describe the propagation and slight dispersion of the almost-monochromatic wavetrain and can be readily computed from the linear dispersion relation; the nonlinear term is due to the self-interaction of the wave packet and represents the leading-order nonlinear correction to the linear dispersion relation.

Since the nonlinear Schrödinger equation does not involve any dependence on the offshore coordinate, it is clear that the envelope of a travelling edge-wave packet at the shoreline should be governed by the same equation. Similarly, it is anticipated that, in the case of two edge waves travelling in opposite directions and interacting with an incident and reflected wavetrain, the edge-wave envelopes A_+ and A_- should satisfy equations of the form

$$A_{\pm T} \pm c_g A_{\pm Z} = \tfrac{1}{2}\mathrm{i}\mu\Omega''\,A_{\pm ZZ} + \frac{\epsilon}{\mu}(\mathrm{i}\,\Omega_2 A_\pm^2\,A_\pm^* + N_1 A_\pm\,A_\mp\,A_\mp^* + N_2 S A_\mp^*) \qquad (X=0,\ Y=0), \tag{12}$$

where, according to (9), $\Omega(k) = (k\sin\alpha)^{\frac{1}{2}}$, and N_1, N_2 are certain constants. The two additional nonlinear terms on the right-hand side of (12) are due respectively to cross-interactions between edge waves travelling in opposite directions and to interactions with the incident wavetrain.

Minzoni & Whitham (1977) derived an evolution equation of the form (12) for the special case of a purely standing edge wave with no longshore modulations ($A_+ = A_-$ and no Z-dependence), and calculated the coefficients Ω_2, N_1, N_2. Since the addition of Z-dependence is not expected to affect the values of these constants, the evolution equations of the edge-wave envelopes at the shoreline are already known, without the need of any further analysis. However, the dependence of the envelopes on the direction out to sea cannot be obtained by (12) alone.

The analysis that follows provides the appropriate evolution equations which describe the behaviour of the envelopes in the offshore direction, and confirms the validity of the results quoted above.

4. Complete evolution equations

When the proposed expansion (8) is substituted into (1)–(3), and terms proportional to $e^{-i\omega t}$ are collected, it is found that, to leading order, S satisfies the inhomogeneous problem

$$S_{xx}+S_{yy}=0 \quad (-x\tan\alpha<y<0), \tag{13}$$

$$S_y-\omega^2 S=4i\omega k^2F^2A_+A_- \quad (y=0), \tag{14}$$

$$S_x\sin\alpha+S_y\cos\alpha=0 \quad (y=-x\tan\alpha). \tag{15}$$

The required solution of (13)–(15) can be expressed as the superposition of a particular solution, representing a purely reflected wave in deep water ($x\rightarrow\infty$), and a solution S_1 of the corresponding homogeneous problem, representing the incident and reflected wavetrain in the absence of edge waves:

$$S=S_1(x,y)+4i\omega A_+A_-P(x,y), \tag{16}$$

where

$$P_{xx}+P_{yy}=0 \quad (-x\tan\alpha<y<0), \tag{17}$$

$$P_y-\omega^2P=k^2\,e^{-2kx\cos\alpha} \quad (y=0), \tag{18}$$

$$P_x\sin\alpha+P_y\cos\alpha=0 \quad (y=-x\tan\alpha), \tag{19}$$

$$P\sim Ce^{i\omega^2x} \quad (x\rightarrow\infty). \tag{20}$$

Following Minzoni & Whitham (1977), the solution of (17), subject to (18)–(20), and for beach angles $\alpha=\pi/2N$, is found as an expansion in the eigenfunctions $S_l(x,y)$, with $S_l(0,0)=1$, of the eigenvalue problem

$$S_{xx}+S_{yy}=0 \quad (-x\tan\alpha<y<0), \tag{21}$$

$$S_y-lS=0 \quad (y=0), \tag{22}$$

$$S_x\sin\alpha+S_y\cos\alpha=0 \quad (y=-x\tan\alpha). \tag{23}$$

In particular, it can be shown that

$$P(x,y)=\frac{2kN}{\pi}\int_0^\infty\frac{C_l}{l-\omega^2}S_l(x,y)\,dl, \tag{24}$$

where

$$C_l=k\int_0^\infty S_l(y,0)\,e^{-2kx\cos\alpha}\,dx, \tag{25}$$

and the path of integration in (24) is indented to pass below the pole at $l=\omega^2$, so that the radiation condition (20) is satisfied. (The details of the derivation of (24) can be found in Minzoni & Whitham (1977), and will not be repeated here.)

The dependence of the wave envelopes on the slow scales is specified by collecting terms proportional to $e^{i\theta_+}$ and $e^{i\theta_-}$. First, Laplace's equation must be satisfied:

$$-\cos\alpha A_{\pm X}+\sin\alpha A_{\pm Y}\pm iA_{\pm Z}+\frac{\mu}{k}(A_{\pm XX}+A_{\pm YY}+A_{\pm ZZ})=0$$
$$(-X\tan\alpha<Y<0). \tag{26}$$

Secondly, the nonlinear surface boundary condition (2), and the bottom boundary condition (3) generate terms proportional to $e^{i\theta_+}$ and $e^{i\theta_-}$, which form the right-hand-side terms of the corresponding inhomogeneous problems for the higher-order corrections to the edge waves. In particular, if $F_{\pm}^{(2)}\,\mathrm{e}^{\mathrm{i}\theta_{\pm}}$ denotes the correction to $A_{\pm}\,F\mathrm{e}^{\mathrm{i}\theta_{\pm}}$, $F_{+}^{(2)}$ and $F_{-}^{(2)}$ satisfy the inhomogeneous problems

$$F_{\pm xx}^{(2)}+F_{\pm yy}^{(2)}-k^2F_{\pm}^{(2)}=0 \quad (-x\tan\alpha<y<0), \tag{27}$$

$$F_{\pm y}^{(2)}-\tfrac{1}{4}\omega^2F_{\pm}^{(2)}=R_{1\pm} \quad (y=0), \tag{28}$$

$$F_{\pm x}^{(2)}\sin\alpha+F_{\pm y}^{(2)}\cos\alpha=R_{2\pm} \quad (y=-x\tan\alpha), \tag{29}$$

where

$$\begin{aligned}R_{1\pm} &= \mu\epsilon^{-\frac{1}{2}}F(\mathrm{i}\omega A_{\pm T}-A_{\pm Y}-\mu A_{\pm TT})\\ &\quad+\mathrm{i}\omega\epsilon^{\frac{1}{2}}A_{\mp}^{*}\{S_xF_x+S_yF_y+\tfrac{1}{2}[F(S_y-\omega^2S)]_y\}\\ &\quad-4\epsilon^{\frac{1}{2}}k^4F^3A_{\pm}\{A_{\pm}A_{\pm}^{*}+2A_{\mp}A_{\mp}^{*}+6\sin^2\alpha\,A_{\mp}A_{\mp}^{*}\} \quad (Y=0),\end{aligned} \tag{30}$$

$$R_{2\pm}=-\mu\epsilon^{-\frac{1}{2}}F(\sin\alpha\,A_{\pm X}+\cos\alpha\,A_{\pm Y}) \quad (Y=-X\tan\alpha). \tag{31}$$

The appropriate evolution equations for the envelopes A_{+} and A_{-} are derived by requiring that no secular terms, which would destroy the assumed uniform validity of the expansion (8), should appear. Accordingly, the inhomogeneous problems (27)–(29) must admit solutions that behave in an acceptable way at infinity. Since the corresponding homogeneous problem has a solution (the Stokes edge wave), the inhomogeneous problem is not soluble, unless $R_{1\pm}$, $R_{2\pm}$ satisfy the appropriate solvability conditions:

$$\int_0^{\infty}R_{1\pm}\,\mathrm{e}^{-kx\cos\alpha}\,\mathrm{d}x-\int_0^{\infty}R_{2\pm}\,\mathrm{e}^{-kr}\,\mathrm{d}r=0, \tag{32}$$

which can be derived by applying Green's theorem in the wedge $-x\tan\alpha<y<0$.

As noted by Whitham (1976), there is a further source of non-uniformity in the solution of the inhomogeneous problems (27)–(29), since

$$R_{1\pm}\sim\mu\epsilon^{-\frac{1}{2}}\mathrm{e}^{-kx\cos\alpha}(\mathrm{i}\omega A_{\pm T}-A_{\pm Y}-\mu A_{\pm TT})+\mathrm{i}\omega\epsilon^{\frac{1}{2}}A_{\mp}^{*}(S_xF_x+S_yF_y) \quad (y=0,\ x\to\infty); \tag{33}$$

in the limit $x\to\infty$, $F_{+}^{(2)}$ and $F_{-}^{(2)}$ are given asymptotically as the solutions of the reduced inhomogeneous problems, obtained by using the asymptotic result (33) as the right-hand side in (28), and neglecting the bottom condition (29). However, owing to the term proportional to $\mathrm{e}^{-kx\cos\alpha}$ in (33), the solutions of the reduced problems exhibit secular behaviour. Therefore, in order to keep (8) uniformly valid, we must insist that

$$A_{\pm Y}-\mathrm{i}\omega A_{\pm T}+\mu A_{\pm TT}=0 \quad (Y=0). \tag{34}$$

Finally, after some manipulation, using (34), and with the aid of (13) and (14), the solvability conditions (32) reduce to

$$\begin{aligned}&\frac{\mu}{2k}(\sin\alpha\,A_{\pm X}+\cos\alpha\,A_{\pm Y})-\mathrm{i}\epsilon\omega k^2A_{\mp}^{*}\\ &\quad\times\left\{4\int_0^{\infty}S_1(x,y=0)F^2(x,y=0)\,\mathrm{d}x-\mathrm{i}\omega\frac{(1+3\cos^2\alpha)}{\sin 2\alpha}A_{\pm}A_{\mp}\right\}\\ &\quad-\epsilon\frac{k^3}{\cos\alpha}\{A_{\pm}(A_{\pm}A_{\pm}^{*}+2A_{\mp}A_{\mp}^{*})-6\sin^2\alpha\,A_{\pm}A_{\mp}A_{\mp}^{*}\}=0 \quad (X=0,\ Y=0).\end{aligned} \tag{35}$$

It should be noted that, in terms of the envelope variables, the conditions (35) hold at the shoreline: $X=0,\ Y=0,\ -\infty<Z<\infty$.

The envelope equations (35) can be further simplified using (26) and (34) so that equations relating A_+, A_- and their derivatives with respect to Z and T, evaluated at the shoreline, eventually emerge; with further use of (16) and (24), these take the form

$$A_{\pm T} \pm \frac{\sin\alpha}{\omega} A_{\pm Z} = -\frac{\mathrm{i}\mu}{2}\frac{2\sin^2\alpha}{\omega^3} A_{\pm ZZ} + \frac{\epsilon}{\mu}\left\{8k^2\cos\alpha\,\chi(\alpha)\,S_0 A_\mp^* - 2\mathrm{i}\frac{k^4}{\omega} A_\pm^2 A_\pm^* + \delta A_\pm A_\mp A_\mp^*\right\} \quad (X = 0,\ Y = 0), \tag{36}$$

where

$$S_0 = S_1(0,0), \tag{37}$$

$$\chi(\alpha)\,S_0 = k\int_0^\infty S_1(x, y=0)\exp(-2kx\cos\alpha)\,\mathrm{d}x, \tag{38}$$

$$\delta = \frac{4k^4}{\omega}\left\{-32N\chi^2\sin 2\alpha + \mathrm{i}\left(3 + \frac{32N\sin 2\alpha}{\pi}\int_0^\infty \frac{C_l^2}{l-\omega^2}\,\mathrm{d}l\right)\right\}, \tag{39}$$

and the integral in (39) is to be interpreted as a principle value.

Therefore the set of equations governing the temporal and spatial evolution of the edge-wave envelopes consists of (26), (34) and (36); the nonlinear equations (36) determine the behaviour of A_+ and A_- at the shoreline, and thus $A_\pm(X=0,\ Y=0, Z, T)$, together with (34), serve as boundary conditions for the determination of $A_\pm(X, Y, Z, T)$ through the linear equations (26).

It is interesting to note that the edge-wave linear dispersion relation $\Omega(k) = (k\sin\alpha)^{\frac{1}{2}}$ implies that

$$c_g \equiv \Omega'(k) = \frac{\sin\alpha}{2\Omega}, \quad \Omega'' = -\frac{c_g^2}{\Omega} = -\frac{\sin^2\alpha}{4\Omega^3}. \tag{40}$$

Recalling that $\Omega = \frac{1}{2}\omega$, it is clear that (36) are precisely of the form (12), suggested by the more intuitive arguments of §3. Furthermore, the evolution equation derived by Minzoni & Whitham (1977) (see equation (63) in their paper†) can be immediately obtained from (36) by letting $A \equiv A_+ = A_-$, $\mu = \epsilon$, and neglecting the dependence on Z:

$$A_T = 8\,k^2\cos\alpha\,\chi(\alpha)\,S_0 A^* + \left(\delta - 2\mathrm{i}\frac{k^4}{\omega}\right)A^2A^*. \tag{41}$$

Thus (36) agree with the already known results at the shoreline.

5. Edge-wave envelope solitons

The evolution equations for the envelope of a single free travelling edge-wave packet are obtained by setting $A_- = S_0 = 0$ in (26), (34) and (36). In addition, the balance $\mu = \epsilon^{\frac{1}{2}}$ is adopted, so that the nonlinear effects enter at the same level as the dispersive effects; with the notational simplification $A \equiv A_+$, the appropriate set of envelope equations reads

$$A_T + c_g A_Z + \mathrm{i}\mu\left(\frac{c_g^2}{2\Omega}A_{ZZ} + \frac{k^4}{\Omega}A^2A^*\right) = 0 \quad (X = 0,\ Y = 0), \tag{42}$$

$$-\cos\alpha\,A_X + \sin\alpha\,A_Y + \mathrm{i}A_Z + \frac{\mu}{k}\left(A_{XX} + A_{YY} + A_{ZZ}\right) = 0 \quad (-X\tan\alpha < Y < 0), \tag{43}$$

$$A_Y - 2\mathrm{i}\Omega A_T + \mu A_{TT} = 0 \quad (Y = 0); \tag{44}$$

(42) is immediately recognized as the nonlinear Schrödinger equation.

† The results of the dimensionless formulation used in this paper can be related to those of Minzoni & Whitham (1977) by setting $g = 1$, and letting $A_+ = A_- \to B^*/4\omega$, $S \to \phi^*/2\omega$, $S_0 \to a_0/2\omega$, where $\omega^2 = 4k\sin\alpha$.

It is convenient to adopt a frame of reference moving with the group velocity and define a new slow timescale:

$$\tilde{Z} = Z - c_g T, \quad \tilde{T} = \mu T, \tag{45}$$

so that, to the same order of approximation, (42)–(44) imply

$$A_{\tilde{T}} + \mathrm{i}\left(\frac{c_g^2}{2\Omega} A_{\tilde{Z}\tilde{Z}} + \frac{k^4}{\Omega} A^2 A^*\right) = 0 \quad (X = 0,\ Y = 0), \tag{46}$$

$$-\cos\alpha\, A_X + \sin\alpha\, A_Y + \mathrm{i}\, A_{\tilde{Z}} + \frac{\mu}{k}(A_{XX} + A_{YY} + A_{\tilde{Z}\tilde{Z}}) = 0 \quad (-X\tan\alpha < Y < 0), \tag{47}$$

$$A_Y + \mathrm{i}\sin\alpha\, A_{\tilde{Z}} + \mu\,(c_g^2\, A_{\tilde{Z}\tilde{Z}} - 2\mathrm{i}\Omega\, A_{\tilde{T}}) = 0 \quad (Y = 0). \tag{48}$$

A special solution of (46)–(48) represents a uniform, periodic, finite-amplitude edge-wave train:†

$$A = A_0 \exp\left\{\frac{2\mu k^4 |A_0|^2}{\cos\alpha}(X\sin\alpha + Y\cos\alpha)\right\}\exp\left(-\mathrm{i}\frac{k^4}{\Omega}|A_0|^2\tilde{T}\right), \tag{49}$$

where $A_0 = A(X = 0,\ Y = 0, \tilde{T} = 0)$. The linear stability of the finite-amplitude edge wave is readily determined: since the coefficients of $A_{\tilde{Z}\tilde{Z}}$ and A^2A^* in the nonlinear Schrödinger equation are of the same sign, linear perturbations to the special solution (49) tend to grow (Benney & Newell 1967), in accordance with the results of Whitham (1976). In fact, the complete solution of the nonlinear Schrödinger equation (Zakharov & Shabat 1972) indicates that the instability gives rise to a series of envelope solitons. The present formulation can be used to investigate the three-dimensional structure of edge-wave envelope solitons.

As is well-known, (46) admits soliton solutions with a sech-type structure:

$$A(X = 0,\ Y = 0, \tilde{Z}, \tilde{T}) = r(\xi)\,\mathrm{e}^{\mathrm{i}(\rho\tilde{Z} + \sigma\tilde{T})}, \quad \xi = \kappa\tilde{Z} - \lambda\tilde{T}, \tag{50}$$

where

$$r(\xi) = \left(\frac{2\zeta}{\nu}\right)^{\frac{1}{2}} \operatorname{sech} \zeta^{\frac{1}{2}}\xi, \tag{51}$$

$$\zeta = \frac{\beta\rho^2 - \sigma}{\beta\kappa^2}, \quad \nu = \frac{\gamma}{\beta\kappa^2}, \tag{52}$$

with

$$\rho = -\frac{\lambda}{2\kappa\beta}, \quad \beta = \frac{c_g^2}{2\Omega}, \quad \gamma = \frac{k^4}{\Omega}. \tag{53}$$

The structure of an edge-wave envelope soliton in the vertical and seaward directions is determined by solving the linear boundary-value problem, consisting of (47) subject to the conditions (48) and (50), in the wedge $-X\tan\alpha < Y < 0$. To leading order,‡ the solution is found to be

$$A(X, Y, \tilde{Z}, \tilde{T}) = r\{\xi + \mathrm{i}\kappa(X\cos\alpha - Y\sin\alpha)\} \times \exp\{\mathrm{i}(\rho\tilde{Z} + \sigma\tilde{T}) - \rho(X\cos\alpha - Y\sin\alpha)\}. \tag{54}$$

It should be remarked that (since $\rho < 0$) the decay rate in the seaward direction of an edge-wave packet with an envelope of the above form is diminished owing to nonlinear effects. In addition, the phase of an envelope soliton depends on the seaward and vertical coordinates.

† This solution has been found before by Whitham (1976) by expanding the frequency and the decay rates of an edge wave in the offshore and vertical directions in powers of the amplitude.

‡ The solution (54) satisfies the condition (48) to $O(\mu)$; the $O(\mu)$ terms in (48) would give rise to higher-order-scale dependencies, which are not considered here. However, it should be noted that it is *not* consistent to drop such terms in (26) and (34) because they contribute to the $O(\mu)$ terms in (36).

6. Instability of standing edge waves

A uniform periodic wavetrain of frequency ω, normally incident and reflected on a beach, is unstable to standing edge-wave perturbations of frequency $\frac{1}{2}\omega$ (Guza & Davis 1974; Minzoni & Whitham 1977). The growth due to the instability is initially exponential, but, as the edge-wave amplitude increases, the nonlinear terms due to edge-wave self-interactions become important, and eventually a steady state is reached. The stability of the resulting steady-state subharmonic standing edge waves to large-scale variations can be investigated, using the already-developed envelope equations.

The steady-state standing finite-amplitude edge wave $A_+ = A_- \equiv A = A_r + iA_i$ is a special solution of (26), (34) and (36):

$$A = \left(\frac{8k^2\omega\chi S_0 \cos\alpha}{2ik^4 - \omega\delta}\right)^{\frac{1}{2}}. \tag{55}$$

The linear stability of A is investigated by writing

$$A_\pm(X, Y, Z, T) = A + a_\pm(X, Y, Z, T), \tag{56}$$

and linearizing for the perturbations $a_\pm$. Thus (36) leads to four linear equations for the real and imaginary parts of $a_\pm = a_{\pm r} + i\,a_{\pm i}$ at the shoreline:

$$\begin{aligned}\frac{\partial a_{\pm r}}{\partial T} \pm c_g \frac{\partial a_{\pm r}}{\partial Z} = \mu \frac{\sin^2\alpha}{\omega^3}\frac{\partial^2 a_{\pm i}}{\partial Z^2} + \frac{\epsilon}{\mu}\{&bS_0 a_{\mp r} + d[|A|^2 a_{\pm i} \\ &+ 2A_i(A_r a_{\pm r} + A_i a_{\pm i})] + \delta_r[|A|^2 a_{\pm r} + 2A_r(A_r a_{\mp r} + A_i a_{\mp i})] \\ &- \delta_i[|A|^2 a_{\pm i} + 2A_i(A_r a_{\mp r} + A_i a_{\mp i})]\},\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{\partial a_{\pm i}}{\partial T} \pm c_g \frac{\partial a_{\pm i}}{\partial Z} = -\mu \frac{\sin^2\alpha}{\omega^3}\frac{\partial^2 a_{\pm r}}{\partial Z^2} + \frac{\epsilon}{\mu}\{&-bS_0 a_{\mp i} - d[|A|^2 a_{\pm r} \\ &+ 2A_r(A_r a_{\pm r} + A_i a_{\pm i})] + \delta_r[|A|^2 a_{\pm i} + 2A_i(A_r a_{\mp r} + A_i a_{\mp i})] \\ &+ \delta_i[|A|^2 a_{\pm r} + 2A_r(A_r a_{\mp r} + A_i a_{\mp i})]\},\end{aligned} \tag{58}$$

where
$$b = 8k^2\chi\cos\alpha, \quad d = \frac{2k^4}{\omega}, \quad \delta = \delta_r + i\delta_i.$$

The stability properties of the steady-state solution A depend on the relative importance of the nonlinear and dispersive effects. Accordingly, the consequences of the balance $\mu = \epsilon^{\frac{1}{2}}$ are investigated first. As is customary in linear stability analysis, it is assumed that

$$(a_{+r}, a_{+i}, a_{-r}, a_{-i}) = (a_1, a_2, a_3, a_4)\,e^{iKZ+pT}, \tag{59}$$

where K is taken to be real. Upon substitution of (59) into (57) and (58), a linear system of four homogeneous algebraic equations for the perturbation amplitudes a_1, a_2, a_3, a_4 is obtained. The solvability condition for this system determines the appropriate values of p:

$$p = \pm ic_g K + \mu q + O(\mu^2), \tag{60}$$

where

$$q^2 - 2\delta_r|A|^2 q + \left(d|A|^2 + 2dA_i^2 - \delta_i|A|^2 - \frac{c_g^2 K^2}{\omega}\right)\left(d|A|^2 + 2dA_r^2 - \delta_i|A|^2 - \frac{c_g^2 K^2}{\omega}\right) = 0. \tag{61}$$

As expected, to leading order, edge-wave envelope perturbations propagate with the group velocity; furthermore, (61) implies that p has a negative real part, since, according to (39), $\delta_r < 0$, and the constant term in (61) is positive. Therefore, if $\mu = \epsilon^{\frac{1}{2}}$, the steady-state edge waves exhibit no instability.

The balance $\mu = \epsilon$ in the perturbation equations (57) and (58) is considered next. Following the usual procedure, substitution of (59) into (57) and (58) leads to a fourth-degree algebraic equation for p. The appropriate values of p depend on K, S_0, ω and the beach angle $\alpha = \pi/2N$. Accordingly, χ and δ must be evaluated as functions of the beach angle, requiring tedius calculations owing to the principal-value integral involved in (39).† For this reason, attention is focused on the stability properties of steady-state edge waves on a mildly sloping beach, $\alpha = \pi/2N \ll 1$, which is usually the case encountered in practice.

As shown by Minzoni & Whitham (1977), in the small-beach-angle limit, χ and δ can be evaluated asymptotically:

$$\chi(\alpha) \sim \frac{1}{2e^2}, \quad \delta(\alpha) \sim \frac{k^4}{\omega}(-1.8413+\mathrm{i}\,0.4942) \quad (\alpha \rightarrow 0). \tag{62}$$

With the change of variables

$$A_\pm = \sin^2\alpha\, A'_\pm, \quad S_0 = \sin^2\alpha\, S'_0, \quad Z = \sin\alpha\, Z', \tag{63}$$

the steady-state edge waves are given by $A'_+ = A'_- \equiv A'$, where A' follows directly from (55). Similarly, the linearized envelope equations for small perturbations $a'_\pm = a'_{\pm\mathrm{r}} + \mathrm{i}a'_{\pm\mathrm{i}}$ to $A' = A'_\mathrm{r} + \mathrm{i}A'_\mathrm{i}$ follow immediately from (57) and (58); when the primes are dropped, the appropriate perturbation equations take the form (57) and (58) with $\epsilon = \mu$, and

$$c_\mathrm{g} = \frac{1}{\omega}, \quad b = \frac{\omega^4}{4\mathrm{e}^2}, \quad d = \frac{\omega^7}{128}, \quad \delta = \omega^7(0.72\times 10^{-2} + \mathrm{i}\,0.19\times 10^{-2}). \tag{64}$$

For perturbations of the form of (59), the linearized equations (57) and (58) specify p through the vanishing of the determinant of the 4×4 matrix $\boldsymbol{B}$ with elements

$$\begin{aligned}
&B_{11} = p + \frac{\mathrm{i}K}{\omega} - 2dA_\mathrm{r}A_\mathrm{i} - \delta_\mathrm{r}|A|^2, \quad B_{12} = -d|A|^2 - 2dA_\mathrm{i}^2 + \delta|A|^2,\\
&B_{13} = -bS_0 - 2\delta_\mathrm{r}A_\mathrm{r}^2 + 2\delta_\mathrm{i}A_\mathrm{r}A_\mathrm{i}, \quad B_{14} = -2\delta_\mathrm{r}A_\mathrm{r}A_\mathrm{i} + 2\delta_\mathrm{i}A_\mathrm{i}^2,\\
&B_{21} = d|A|^2 + 2dA_\mathrm{r}^2 - \delta_\mathrm{i}|A|^2, \quad B_{22} = p + \frac{\mathrm{i}K}{\omega} + 2dA_\mathrm{r}A_\mathrm{i} - \delta_\mathrm{r}|A|^2,\\
&B_{23} = -2\delta_\mathrm{r}A_\mathrm{r}A_\mathrm{i} - 2\delta_\mathrm{i}A_\mathrm{r}^2, \quad B_{24} = bS_0 - 2\delta_\mathrm{r}A_\mathrm{i}^2 - 2\delta_\mathrm{i}A_\mathrm{r}A_\mathrm{i},\\
&B_{31} = B_{13}, \quad B_{32} = B_{14}, \quad B_{33} = p - \frac{\mathrm{i}K}{\omega} - 2dA_\mathrm{r}A_\mathrm{i} - \delta_\mathrm{r}|A|^2, \quad B_{34} = B_{12},\\
&B_{41} = B_{23}, \quad B_{42} = B_{24}, \quad B_{43} = B_{21}, \quad B_{44} = p - \frac{\mathrm{i}K}{\omega} + 2dA_\mathrm{r}A_\mathrm{i} - \delta_\mathrm{r}|A|^2.
\end{aligned} \tag{65}$$

As already indicated, the values of p are the roots of a fourth-degree algebraic equation, and, from (65), it is clear that they are either real or occur in complex conjugate pairs. The values of p were computed by solving the appropriate algebraic equation numerically using Newton's method. Since ω is the (dimensionless) frequency of the incoming wavetrain, it can be normalized to unity without any loss of generality. The four values of p were calculated as functions of K for $S_0 = 1$. As expected, for $K = 0$, the equilibrium state is stable: two of the roots have negative

† Minzoni & Whitham (1977) give values of the principal-value integral for $\alpha = \frac{1}{4}\pi, \frac{1}{6}\pi$.

real parts, while the other two are pure imaginary. However, when K is different from zero, the pure imaginary pair of roots can become complex with a positive real part, implying that such disturbances are unstable. In particular, there is a certain range of $K (0 < K < 0.052)$ for which instability is present; the growth rate (real part of p), corresponding to the most unstable disturbance ($K = 0.036$), is found to be 0.0163. It should be noted that the range of unstable perturbation wavenumbers for different values of S_0 follows directly from the results for $S_0 = 1$, since (55) and (65) imply that $p(mK, mS_0) = mp(K, S_0)$ for any m. Thus the most-unstable wavenumber and the maximum growth rate are linearly increasing functions of S_0. Furthermore, the corresponding imaginary parts of p turn out to be substantially smaller than the real parts, so that the unstable disturbances represent nearly standing waves of growing amplitude.

Recalling the adopted scalings (10) and (63), the unstable perturbation wavelengths Λ can be related to the wavelength λ_e, of the steady-state edge waves, (and to the wavelength, $\lambda_\infty = \lambda_e/4 \sin\alpha$ of the incoming wave in deep water):

$$\Lambda = \frac{\sin^2\alpha}{8\pi\epsilon S_0 K}\lambda_e \quad (0 < K < 0.052), \tag{66}$$

where the small parameter $\epsilon = a_\infty/\lambda_\infty$, is the ratio of the amplitude to the wavelength of the incoming wave in deep water and S_0 is defined as in (37). Expressing the amplitude a_∞, of the incoming wave at infinity in terms of the amplitude a_0, at the shoreline (Minzoni & Whitham 1977),

$$a_\infty = \left(\frac{2\alpha}{\pi}\right)^{\frac{1}{2}} a_0, \tag{67}$$

(66) can be rewritten in the form:

$$\Lambda = \left(\frac{\alpha}{2\pi^3}\right)^{\frac{1}{2}} \frac{\sin^2\alpha}{4\epsilon K}\lambda_e \quad (0 < K < 0.052). \tag{68}$$

Similarly, the growth rate G of the most-unstable perturbation is given in terms of the period τ, of the incoming wave by

$$G = 0.0163\left(\frac{2\pi^3}{\alpha}\right)^{\frac{1}{2}} \frac{2\pi\epsilon}{\tau\sin^2\alpha}. \tag{69}$$

Table 1 gives the values of the most-unstable perturbation wavelengths and the corresponding growth rates for various wave amplitudes a_0; having in mind the experiments of Guza & Inman (1975) and Guza & Bowen (1976), the typical values $a = 6°$, $\tau = 2.7$ s were used in the calculations. It is clear that the typical scale of the unstable disturbances is relatively long compared with the edge-wave wavelength, and can be comparable to the wavelength of the incoming wave in deep water. This suggests why no instability was observed by Guza & Inman (1975) and Guza & Bowen

a_0(cm)	Λ/λ_e	$G\tau$
1.8	7.56	0.094
2.0	6.80	0.104
2.5	5.44	0.130
3.0	4.53	0.156

TABLE 1. Wavelengths and growth rates of the most-unstable modulations for various incident-wave amplitudes at the shoreline; $\alpha = 6°$, $\lambda_e = 4.75$ m, $\tau = 2.7$ s

(1976), who used beaches of limited width. Furthermore, since the laboratory beaches were confined within rigid walls, the end conditions restricted the possible longshore modulations. For a conclusive verification of the predicted instability, experiments with wider beaches, where end conditions are less important, are desirable.

The present theory is based on a number of idealizations: inviscid, small-amplitude, non-breaking disturbances on perfectly reflective beaches are assumed; such conditions are rarely met on natural beaches. Nevertheless, one could speculate that the instability of standing subharmonic edge waves to large-scale variations could be of some importance for certain observed phenomena. In fact, the field observations of Munk (1949) and Tucker (1950) indicate that long standing waves can exist on beaches. As noted by Foda & Mei (1981), such long-period oscillations cannot be attributed to the instability of incident waves to subharmonic edge-wave perturbations, since the observed oscillations are of much longer period than that of typical waves in the ocean. However, the unstable edge-wave modulations evolve on a slow timescale compared with the period of the incoming waves (see table 1). Accordingly, the modulational instabilities of subharmonic edge waves could contribute to the generation of long-period oscillations near the shore, in addition to the wave-interaction mechanisms of Gallagher (1971) and Foda & Mei (1981). Finally, the longshore variations, produced by the unstable edge-wave modulations, could play some role in the excitation of the observed widely spaced rip currents and circulation cells on natural beaches.

REFERENCES

Benney, D. J. & Newell, A. C. 1967 *J. Maths & Phys.* **46**, 133–139.

Bowen, A. J. & Inman, D. L. 1969 *J. Geophys. Res.* **74**, 5479–5490.

Foda, M. A. & Mei, C. C. 1981 *J. Fluid Mech.* **111**, 319–345.

Gallagher, B. 1971 *J. Fluid Mech.* **49**, 1–20.

Galvin, C. J. 1965 *Trans. Am. Geophys. Union* **46**, 112.

Greenspan, H. P. 1956 *J. Fluid Mech.* **1**, 574–592.

Guza, R. T. & Bowen, A. J. 1976 *J. Mar. Res.* **34**, 269–293.

Guza, R. T. & Davis, R. E. 1974 *J. Geophys. Res.* **79**, 1285–1291.

Guza, R. T. & Inman, D. L. 1975 *J. Geophys. Res.* **80**, 2997–3011.

Minzoni, A. A. & Whitham, G. B. 1977 *J. Fluid Mech.* **79**, 273–287.

Munk, W. H. 1949 *Trans. Am. Geophys. Union* **30**, 849–854.

Munk, W. H., Snodgrass, F. & Carrier, G. F. 1956 *Science* **123**, 127–132.

Rockliff, N. 1978 *Math. Proc. Camb. Phil. Soc.* **83**, 463–479.

Stokes, C. G. 1846 In *Proc. 16th Meet. Brit. Assn Adv. Sci., Southampton*, pp. 1–20. John Murray.

Tucker, M. J. 1950 *Proc. R. Soc. Lond.* **A202**, 565–573.

Whitham, G. B. 1976 *J. Fluid Mech.* **74**, 353–368.

Whitham, G. B. 1979 *Lectures on Wave Propagation, Tata Institute of Fundamental Research, Bombay*. Springer.

Zakharov, V. E. & Shabat, A. B. 1972 *Sov. Phys. JETP* **34**, 62.